

# Dust of Dark Energy

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We introduce a novel class of field theories where energy always flows along timelike geodesics, mimicking in that respect dust, yet which possess non-zero pressure. This theory comprises two scalar fields, one of which is a Lagrange multiplier enforcing a constraint between the other’s field value and derivative. We show that this system possesses no wave-like modes but retains a single dynamical degree of freedom. Thus, the sound speed is always identically zero on all backgrounds. In particular, cosmological perturbations reproduce the standard behaviour for hydrodynamics in the limit of vanishing sound speed. Using all these properties we propose a model unifying Dark Matter and Dark Energy in a single degree of freedom. In a certain limit this model exactly reproduces the evolution history of  $\Lambda$ CDM, while deviations away from the standard expansion history produce a potentially measurable difference in the evolution of structure.

## I. DUSTY FLUID WITH PRESSURE?

How can one obtain dust from a scalar field? One can imagine a canonical scalar-field where the kinetic term is constrained to be equal to the potential. We can implement this property by introducing a Lagrange multiplier,  $\lambda$ , in the Lagrangian,

$$\mathcal{L} = \lambda \left( \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right).$$

We then find that the pressure is identically vanishing on all solutions and energy follows geodesics. This model describes the usual dust without vorticity.

How can we obtain “dust with pressure”? We can generalise the above by adding some function of the scalar field and its derivatives to the Lagrangian,

$$\mathcal{L} = K(\varphi, \partial\varphi) + \lambda \left( \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right).$$

The constraint remains in effect and standard scalar-field dynamics are not restored. In fact, we will show that fluid elements in all such theories *also* always flow along geodesics, mimicking in that respect standard dust, yet the fluid has non-vanishing pressure. With this simple idea we have separated the notion that the pressure of the fluid is tied to the motion of a fluid element as is the situation in the usual case, e.g. radiation or cold dark matter. A parcel of such fluid will flow along geodesics, yet a manometer will record a pressure changing with time.

In this paper, we introduce this new class of scalar-field models, which we will call  $\lambda\varphi$ -*fluids*. These theories are described by an action containing two scalar fields,  $\varphi$  and

$\lambda$ , where the latter plays the role of a Lagrange multiplier and enforces a constraint relating the value of the scalar field  $\varphi$  to the norm of its derivative. This constraint forces the dynamics of the  $\lambda\varphi$ -*fluid* to be driven by a system of two first-order *ordinary* differential equations, one for the field  $\varphi$ , the other for the Lagrange multiplier. As a consequence, there are no propagating wave-like degrees of freedom and the sound speed for perturbations is exactly zero irrespective of the background solution. However, the initial-value problem still requires the specification of two functions on the initial time slice. Thus, effectively, a single dynamical degree of freedom remains.

Provided that the derivatives of the scalar field  $\varphi$  are time-like, the system can be interpreted as a perfect fluid. However, for a  $\lambda\varphi$ -*fluid* given by a particular action, the relation between the pressure  $p$  and the energy density  $\varepsilon$  is solution dependent. We show that an arbitrary effective equation of state, including phantom ones, can be obtained by choosing the form of the Lagrangian appropriately. In addition, for all  $\lambda\varphi$ -*fluids*, there always exists a region in their phase spaces in which the  $\lambda\varphi$ -*fluid* is effectively pressureless. We will exploit this feature to model the evolution of the cosmological background from matter domination through to the acceleration era as being driven by the dynamics of a single degree of freedom provided by the  $\lambda\varphi$ -*fluid*.

The key novel aspect of this class of theories is that the four-acceleration is always zero, even when the pressure does not vanish. This is a result of the constraint’s eliminating those fluid configurations where the pressure has a gradient orthogonal to the fluid velocity.

Motivated by these properties, we will use the  $\lambda\varphi$ -*fluid* to frame the problem of the dark sector in cosmology in a unified manner. The existence of the dark sector in the Universe’s energy budget is now established beyond reasonable doubt. The standard model of cosmology,  $\Lambda$ CDM, splits it into two constituents: cold dark matter (“CDM”)—a pressureless fluid (“dust”) which clusters allowing baryonic structures to form in its potential wells and detectable to this day in the form of halos around galaxies—and dark energy (“DE”—a form of

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energy that appears to be smooth and to have an equation of state close to a cosmological constant. This dichotomy of phenomenology has made it difficult to build a compelling model which would treat the two dark components in a unified fashion. Nonetheless, some models exist in the literature: [1–13]. It is not hard to see that, given a fluid which clusters like dust yet has arbitrary pressure, we can construct such a unified model—which we call Dusty Dark Energy (“DDE”). We present the main results of our application of  $\lambda\varphi$ -fluid to such a cosmology beneath. We refer the reader to the full analysis in the main body of the paper, section V.

### A. Summary of Cosmological Results

We have studied cosmological perturbations in the case when an arbitrary  $\lambda\varphi$ -fluid dominates the Universe. We have derived the closed-form equation for the evolution of the Newtonian potential  $\Phi$  which turns out to recover the standard result for general hydrodynamics in the limit of vanishing sound speed. This evolution is determined by background expansion history only and in the limit of the  $\Lambda$ CDM expansion history the evolution of perturbations is exactly as in the standard case. We have also written down an action for perturbations which explicitly shows that there are no ghosts in this theory when the equation of state for the  $\lambda\varphi$ -fluid is non-phantom. We demonstrate that on a classical level our model can cross the phantom divide without singularities while linear perturbations continue to evolve stably; however, the perturbations do become ghosts at this point, hence making the system unstable (in particular quantum-mechanically) when interactions are taken into account. In this paper we put the investigation of that instability and issues related to the possible strong coupling scales aside.

We consider a universe comprising only radiation and the  $\lambda\varphi$ -fluid which will describe both CDM and DE. To illustrate more concretely the phenomenology we have focused on a specific family of models which is parameterised by  $w_{\text{fin}}$ —the equation of state of the  $\lambda\varphi$ -fluid in the asymptotic future. Given that the initial values for the  $\lambda\varphi$ -fluid are chosen appropriately, the radiation becomes subdominant while the  $\lambda\varphi$ -fluid is still far off its final attractor (given by  $w_{\text{fin}}$ ) and evolves approximately like dust, giving an epoch of matter domination. The duration of this epoch is also determined by the initial values.

In the limit  $w_{\text{fin}} \rightarrow -1$  this family of models recovers exactly the background evolution and growth of structure of  $\Lambda$ CDM. However, if  $w_{\text{fin}} \neq -1$ , the evolution of the background and perturbations differs from a “ $w$ CDM” model comprising cold dark matter and dark energy with a constant equation of state. We illustrate the evolution of the effective equation of state for the dark sector in a selection of different  $w_{\text{fin}}$ -cosmologies in Fig. 1. We find that the transition from matter-domination to dark

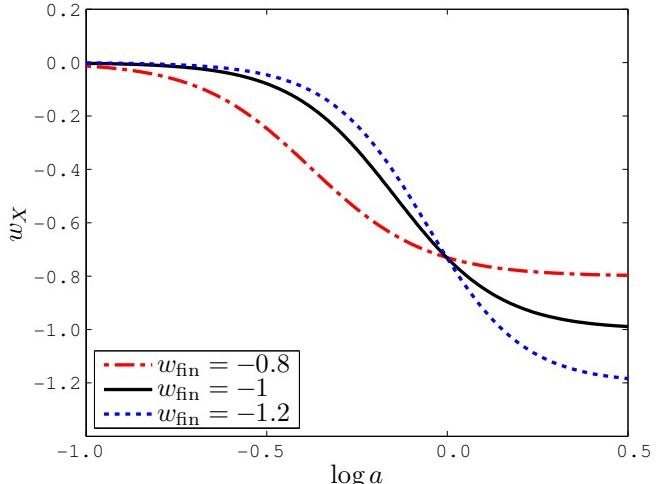


FIG. 1: Time evolution of the total effective equation of state for the dark sector. The black solid line represents the evolution in  $\Lambda$ CDM which is identical to that of the  $w_{\text{fin}} = -1$  model. Models with final equations of state  $1 + w_{\text{fin}} > 0$  begin to deviate from matter domination earlier and the transition is slower than  $\Lambda$ CDM. The opposite is true in phantom models. The evolution is normalised such that the equation of state at  $a = 1$  matches the best-fit result for the  $\Lambda$ CDM cosmology as determined by WMAP7 results,  $w_0 = -0.74$  [14].

energy domination differs from that of  $w$ CDM (Fig. 2). We compare the growth of linear perturbations with that of  $\Lambda$ CDM in Fig. 3. We find that models with  $1 + w_{\text{fin}} > 0$  exhibit a growth factor suppressed by a few tens of percent. Also the Newtonian potential here is lower by a few percent than in  $\Lambda$ CDM, which would decrease the integrated Sachs-Wolfe effect.

### B. Open Questions and Future Directions

Finally, we mention open issues in this setup which remain to be addressed

- The nature of caustics: It is well known that in non-canonical field theories caustic can develop, e.g. see Ref. [15] on caustics in Sen’s string-theoretical tachyon matter [16, 17] and Ref. [18] on caustics in the ghost condensate [19] and the discussion in Refs [20, 21] on caustics in Hořava gravity [22]. We expect that the  $\lambda\varphi$ -fluid will develop caustics. The question is how to interpret the multivalued regions once this occurs.
- Can  $\lambda\varphi$ -fluids virialise? If the  $\lambda\varphi$ -fluid is to really model non-linear structure, it must be able to form static and stable configurations (e.g. halos).
- What is the origin for the initial conditions for Dusty Dark Energy introduced in section V? Could a more generalised setup provide a solution for the

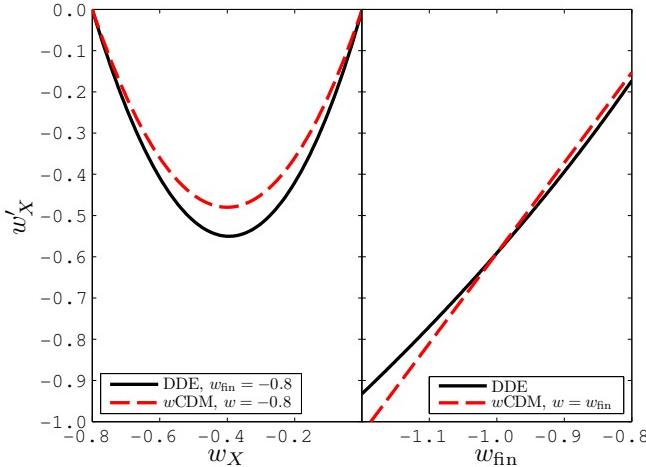


FIG. 2: Comparison of the derivative of the effective equation of state for Dusty Dark Energy (Eq. (41)) with that for a dark matter plus dark energy with a constant equation of state,  $w\text{CDM}$ , (Eq. (43)). The magnitude of the derivative determines the duration of the transition between matter domination and the acceleration era. The left panel shows that for  $w_{\text{fin}} > -1$ , the transition in the DDE model is more rapid than the corresponding  $w\text{CDM}$  model. On the other hand, for phantom  $w_{\text{fin}}$  this transition is slower than for the corresponding  $w\text{CDM}$  model, as shown in the panel on the right.

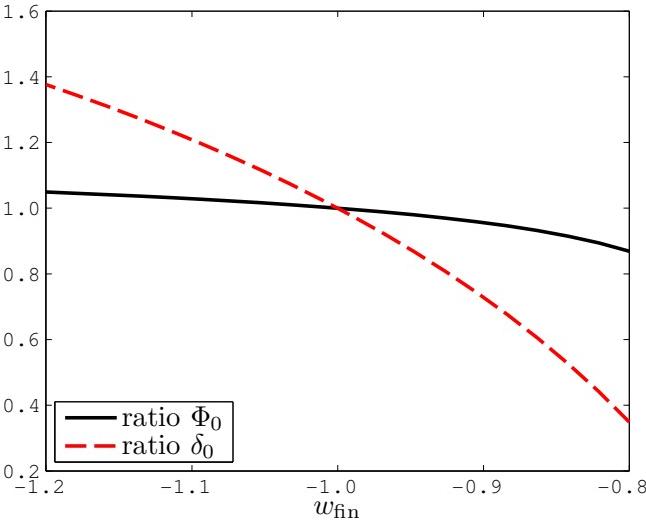


FIG. 3: The comparison of the total growth of perturbation amplitude between Dusty Dark Energy and  $\Lambda\text{CDM}$ . The evolution of the Newtonian potential,  $\Phi$ , is determined by Eq. (25) and deviates by a few percent from its  $\Lambda\text{CDM}$  values by a few percent. This evolution will affect the strength of the ISW signal in the CMB. On the other hand, the evolution of the density perturbation on subhorizon scales is determined by Eq. (22) and is affected much more strongly.

coincidence problem? In our model, we require that at some point during the radiation epoch the energy density of the  $\lambda\varphi$ -fluid be equal to the energy density of CDM: we have no alternative to the standard dark-matter freeze out scenario which would produce this in a natural manner.

- What is the Hamiltonian structure of this theory? How to quantise it? See, for example [23]. Is the structure of this theory stable as a result of radiative corrections: would a kinetic term for  $\lambda$  be generated?
- What is the strong-coupling scale for perturbations? We should note that our model is rather similar to a potentially singular limit of Hořava gravity [22]. There it was found [21] that, contrary to our case, the sound speed for cosmological perturbations becomes imaginary, and that the strong coupling scale may be extremely low.
- What is the rate of instability for the phantom case? Is this instability catastrophic as it is usually for ghost degrees of freedom? It is possible that the absence of propagating wave-like degrees of freedom may change the standard picture [24–28].
- The action formulation of this theory allows us to consider couplings to standard model fields. It is natural, for example, to consider Lorentz violation in this framework in analogy to Einstein aether theories [29–31].
- Can this theory be a low-energy limit of a more fundamental theory?
- Can our conceit with the constraint be usefully extended beyond scalar fields to theories with fermions, vector fields or many degrees of freedom?

## II. DYNAMICS IN GENERAL CURVED SPACE-TIME

Let us consider a scalar field theory given by the action

$$S = \int d^4x \sqrt{-g} \left( K(\varphi, X) + \lambda \left( X - \frac{1}{2}\mu^2(\varphi) \right) \right),$$

where the field  $\lambda$  is a “Lagrange multiplier” and does not have a kinetic term, while

$$X \equiv \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi,$$

is a standard kinetic term for the field  $\varphi$ ,  $K(\varphi, X)$  is an arbitrary function of  $X$  and  $\varphi$ , while  $\mu(\varphi)$  is an arbitrary function of the scalar field  $\varphi$ . In hydrodynamical language,  $\varphi$  is one of the velocity potentials and does not necessarily have to represent a fundamental degree

of freedom;  $\mu(\varphi)$  plays the role of the specific inertial mass [32]. Let us further assume a standard minimal coupling with gravity.

The equations of motion are

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \lambda} = X - \frac{1}{2} \mu^2 (\varphi) = 0, \quad (1)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \varphi} = K_\varphi - \nabla_\alpha (K_X \nabla^\alpha \varphi) - \lambda \mu \mu_\varphi - \nabla_\alpha (\lambda \nabla^\alpha \varphi) = 0. \quad (2)$$

Here and throughout the paper we denote partial derivatives by subscripts. The Lagrange multiplier imposes a non-holonomic constraint<sup>1</sup> on the dynamics of the fields, hence the action cannot be written purely in terms of  $\varphi$ . In the case of time-like derivatives:  $X > 0$ , similarly to k-essence [33–35], we can introduce an effective 4-velocity

$$u_\alpha = \frac{\nabla_\alpha \varphi}{\sqrt{2X}} = \mu^{-1} \nabla_\alpha \varphi, \quad (3)$$

where in the last equality we have used the “constraint” Eq. (1). Further, we can see that the corresponding effective four-acceleration always vanishes,

$$a_\beta = \dot{u}_\beta = (\mu^{-1} \nabla_\gamma \varphi) \nabla^\gamma (\mu^{-1} \nabla_\beta \varphi) = 0. \quad (4)$$

Here and throughout the paper we use the notation  $\dot{(\ )} = u^\alpha \nabla_\alpha (\ )$  for the derivative along  $u^\alpha$ . Thus, on equations of motion,  $u^\alpha$  are tangents to time-like geodesics. It is also convenient to write Eq. (1) in this form

$$\dot{\varphi} = \mu(\varphi). \quad (5)$$

The general solution of Eq. (5) is

$$\varphi(x^\alpha) = f(\tau - \psi(\mathbf{x})), \quad (6)$$

where  $\tau$  is a time parameterising the congruence of geodesics given by  $u^\alpha$ ,  $f$  is a general function solving Eq. (5) and  $\psi(\mathbf{x})$  is an arbitrary function of spatial coordinates  $\mathbf{x}$  in the hypersurface normal to  $u^\alpha$ . One can consider  $\varphi$  to be an *intrinsic clock* for our system, since Eq. (6) defines a time reparameterization.

The energy-momentum tensor (EMT) is

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} = (K_X + \lambda) \nabla_\alpha \varphi \nabla_\beta \varphi - K g_{\alpha\beta},$$

which is of the perfect-fluid form. Using hydrodynamical notation the EMT can be rewritten as

$$T_{\alpha\beta} = (\varepsilon + p) u_\alpha u_\beta - p g_{\alpha\beta},$$

with the energy density given by

$$\varepsilon(\lambda, \varphi) = \mu^2 (K_X + \lambda) - K, \quad (7)$$

<sup>1</sup> As a technical point, the system possesses two second class constraints, one which is primary and the other secondary.

where into  $K_X$  and  $K$  we substitute the constraint  $X = \frac{1}{2} \mu^2$ . The pressure is a function of the intrinsic clock  $\varphi$  only

$$p(\varphi) = K \left( \varphi, \frac{\mu^2(\varphi)}{2} \right). \quad (8)$$

This is the key feature responsible for the absence of acceleration, Eq. (4), since the gradient of the pressure is always parallel to  $u^\alpha$ . This can be explicitly seen by considering the conservation of the EMT:

$$\begin{aligned} \nabla_\alpha T^{\alpha\beta} &= (\dot{\varepsilon} + \dot{p} + \theta(\varepsilon + p)) u^\beta + (\varepsilon + p) a^\beta - \nabla^\beta p = \\ &= (\dot{\varepsilon} + \theta(\varepsilon + p)) u^\beta, \end{aligned}$$

where we have used the fact that pressure does not have gradients orthogonal to  $u^\alpha$  and  $\theta \equiv \nabla_\alpha u^\alpha$  is the expansion of a congruence of geodesics under consideration. For perfect fluids in the usual case, the divergence of the EMT has two components: one parallel to the velocity  $u^\beta$  describing the conservation of energy and one parallel to the four-acceleration  $a^\beta$ . Here the latter is identically zero and so the conservation of the EMT reduces to the conservation of energy,

$$\dot{\varepsilon} + \theta(\varepsilon + p) = 0. \quad (9)$$

Then by choosing  $K$  and  $\mu$  appropriately one can arrange for a general evolving equation of state

$$w_X \equiv \frac{p}{\varepsilon}.$$

Despite this, the energy flux of the  $\lambda\varphi$ -fluid,  $T^{\alpha\beta} u_\beta = \varepsilon u^\alpha$ , always follows time-like geodesics, as is the case for perfect fluids in the absence of pressure. Note that in FRW universes, the energy of an arbitrary fluid flows along time-like geodesics as well, provided that the fluid configuration be bound to respect the homogeneity and isotropy of the FRW spacetime. This is of course not true of inhomogeneous perturbations in the standard case.

Now, let us refocus on the equation of motion. Using the constraint Eq. (1) we can re-express the gradient of the kinetic term and the expansion in terms of  $\varphi$ ,

$$\begin{aligned} \nabla_\alpha X &= \mu \mu_\varphi \nabla_\alpha \varphi, \\ \theta &= \nabla_\alpha u^\alpha = \mu^{-1} \square \varphi - \mu_\varphi. \end{aligned} \quad (10)$$

The system of equations of motion Eq. (2) and Eq. (1) can be written as

$$\dot{\varphi} = \mu(\varphi), \quad (11)$$

$$\dot{\lambda} = -\mu^{-2} (\varepsilon_\varphi \mu + (\varepsilon + p) \theta), \quad (12)$$

where for the partial derivative of  $\varepsilon_\varphi$  we differentiate Eq. (7) to obtain

$$\varepsilon_\varphi = \mu \mu_\varphi (\mu^2 K_{XX} + K_X + 2\lambda) + \mu^2 K_{X\varphi} - K_\varphi. \quad (13)$$

The equations of motion, Eqs (11) and (12), for our system have been reduced in this way to two first-order ordinary differential equations. They then have to be supplemented by the Landau-Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{a\beta}u^\alpha u^\beta,$$

for the expansion  $\theta$  and similar equations for the shear  $\sigma_{\alpha\beta}$  or, equivalently, by Einstein's equations<sup>2</sup>.

From the above system of equations of motion and the 4-velocity Eq. (3) it follows that the Cauchy problem locally has a unique solution depending on two functions  $\varphi(\mathbf{x})$  and  $\lambda(\mathbf{x})$  given on an initial spacelike hypersurface  $\Sigma$ . Moreover, from Eq. (11) and Eq. (12) it follows that the solutions propagate along time-like geodesics. Therefore in the future of the geodesic  $\gamma_i$  starting at  $\mathbf{x}_i$  on  $\Sigma$  the solution only depends on the initial values of  $\varphi(\mathbf{x}_i)$  and  $\lambda(\mathbf{x}_i)$  at the point  $\mathbf{x}_i$ —the sound speed  $c_s$  is identically equal to zero for all configurations of  $\lambda\varphi$ -fluid. This means that there are no wave-like dynamical degrees of freedom. Adding dynamical gravity does not change the picture.

As we have already mentioned in the introduction, the global Cauchy problem may be ill-defined for some initial data since caustics may develop. However, this problem is not unusual for non-canonical field theories.

Finally we note that, since  $\varepsilon_\lambda = \mu^2 \neq 0$ , we could use the pair  $(\varphi, \varepsilon)$  instead of  $(\varphi, \lambda)$  as independent variables. In that case, one has to use energy conservation Eq. (9) instead of Eq. (12) as the second equation of motion.

### III. DYNAMICS IN COSMOLOGY

#### A. Cosmological Background

In the case of a background cosmology of a pure  $\lambda\varphi$ -fluid in the FRW universe with the metric

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2,$$

our equations of motion take the form

$$\begin{aligned} \dot{\varphi} &= \mu(\varphi), \\ \dot{\lambda} &= -\mu^{-2}(\varepsilon_\varphi\mu + 3H(\varepsilon + p)), \end{aligned} \tag{14}$$

where we have used  $\theta = 3H \equiv 3\dot{a}/a$ . Alternatively, we can use energy conservation Eq. (9) instead of the equation for  $\lambda$ . To close this system we use the Friedmann equation<sup>3</sup>

$$H^2 = \frac{\varepsilon}{3M_{\text{Pl}}^2} = \frac{\mu^2(K_X + \lambda) - K}{3M_{\text{Pl}}^2}, \tag{15}$$

<sup>2</sup> Note that the rotation  $\omega_{\alpha\beta}$  is zero because of Eq. (3).

<sup>3</sup> Here, for simplicity, we have assumed a spatially flat case.

where  $M_{\text{Pl}} \equiv (8\pi G_N)^{-1/2}$  is the reduced Planck mass. The phase space for this system of two first-order ordinary differential equations is  $(\lambda, \varphi)$ .

For completeness we also present the second Friedmann equation

$$\dot{H} = -\frac{\varepsilon + p}{2M_{\text{Pl}}^2} = -\frac{\mu^2(K_X + \lambda)}{2M_{\text{Pl}}^2}. \tag{16}$$

The presence of an external energy density of some other cosmological fluid brings with it the corresponding change to the Friedmann equations, Eqs (15), (16) but no changes to the equations of motion apart from those implied by the change in  $H$ .

#### B. Cosmological Perturbations

Symmetry considerations imply that our scalar field at linear order only contributes to the scalar part of the cosmological perturbations. In Newtonian gauge, the metric for scalar perturbations is

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Psi)d\mathbf{x}^2.$$

The absence of anisotropic stress,  $\Phi = \Psi$ , simplifies the perturbed Einstein equations to (see e.g. [36]):

$$\frac{\Delta}{a^2}\Phi - 3H(\dot{\Phi} + H\Phi) = \frac{\delta\varepsilon_{\text{tot}}}{2M_{\text{Pl}}^2}, \tag{17}$$

$$(\dot{\Phi} + H\Phi)_{,i} = \frac{(\varepsilon + p)}{2M_{\text{Pl}}^2}\delta u_{\text{tot}\parallel i}, \tag{18}$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2)\Phi = \frac{\delta p_{\text{tot}}}{2M_{\text{Pl}}^2}, \tag{19}$$

where  $\Delta = \partial_i\partial_i$  and the  $\delta u_{\text{tot}\parallel i}$  is the potential (scalar) part of the  $i$  component of the total four-velocity. Perturbing the equations of motion gives:

$$\delta\dot{\varphi} = \mu_\varphi\delta\varphi + \Phi\mu, \tag{20}$$

for the constraint Eq. (1) and

$$\delta\dot{\varepsilon} - \left(3\dot{\Phi} + \frac{\Delta}{a^2}\left(\frac{\delta\varphi}{\mu}\right)\right)(\varepsilon + p) + 3H(\delta\varepsilon + \delta p) = 0, \tag{21}$$

for the perturbations of the energy density. See Appendix A for the derivation. This equation is standard for hydrodynamical matter [36, p. 312, Eq. (7.105)]. The non-standard input of our model is that  $\delta p = p_\varphi\delta\varphi$  and that Eq. (20) describes the evolution of the velocity potential  $v = \mu^{-1}\delta\varphi$  for time-like geodesics. Note that in the above equations, we have not assumed the domination of the  $\lambda\varphi$ -fluid. Thus, equations (20) and (21) can be used to follow the dynamics of the linear perturbations of  $\lambda\varphi$ -fluid through the entire history of the universe.

For future discussion it is helpful to re-express the above as an equation for the relative perturbation  $\delta_\varepsilon = \delta\varepsilon/\varepsilon$ :

$$\begin{aligned}\dot{\delta}_\varepsilon &= -3H\frac{p_\varphi}{\varepsilon}\delta\varphi + \\ &+ 3Hw_X\delta_\varepsilon + \left(3\dot{\Phi} + \frac{\Delta}{a^2}\left(\frac{\delta\varphi}{\mu}\right)\right)(1+w_X).\end{aligned}\quad (22)$$

Combining the above with the perturbed constraint Eq. (20) and obtaining the Newtonian potential  $\Phi$  through the perturbed Einstein equations (17), (18), (19) closes the system and allows us to describe the evolution of the perturbations in the  $\lambda\varphi$ -fluid in general case.

When the  $\lambda\varphi$ -fluid dominates the energy density and the perturbations, a significant simplification occurs since

$$\begin{aligned}\delta u_{\text{tot}}{}_{||i} &= \mu^{-1}\partial_i\delta\varphi, \\ \delta\varepsilon_{\text{tot}} &= \varepsilon_\varphi\delta\varphi + \mu^2\delta\lambda, \\ \delta p_{\text{tot}} &= p_\varphi\delta\varphi.\end{aligned}$$

Using these expressions for perturbations, we can first integrate Eq. (18) to obtain

$$\delta\varphi = \frac{2M_{\text{Pl}}^2\mu}{\varepsilon + p}\left(\dot{\Phi} + H\Phi\right) = -\frac{\mu}{H}\left(\dot{\Phi} + H\Phi\right),$$

where in the last equality we have used the second Friedmann equation (16). Combining this with Eq. (19), we can obtain a closed expression for  $\Phi$

$$\begin{aligned}\ddot{\Phi} + \dot{\Phi}H\left(4 + \frac{\mu p_\varphi}{2M_{\text{Pl}}^2\dot{H}H}\right) + \\ + \left(2\frac{\dot{H}}{H^2} + 3 + \frac{\mu p_\varphi}{2M_{\text{Pl}}^2\dot{H}H}\right)H^2\Phi = 0.\end{aligned}\quad (23)$$

Thus, the Newtonian potential always evolves as

$$\Phi(t, \mathbf{x}) = f_1(t)C_1(\mathbf{x}) + f_2(t)C_2(\mathbf{x}), \quad (24)$$

where  $C_1(\mathbf{x})$ ,  $C_2(\mathbf{x})$  are arbitrary functions of the spatial coordinates while  $f_1(t)$  and  $f_2(t)$  are solutions of the homogeneous ordinary differential equation Eq. (23). Using the solution Eq. (24) we can find all other quantities. In particular, substituting this solution for  $\Phi$  into the Poisson equation (17), we obtain  $\delta\varepsilon$ . The separation of variables in the solution (24) implies that, as we have discussed above, the sound speed is zero.

It is convenient to introduce dimensionless time (e-folds number)  $N \equiv \ln a$  and rewrite the differential equation for the Newtonian potential  $\Phi$  in terms of  $N$

$$\Phi'' + \Phi'\left(4 + \frac{H'}{H} + \Gamma\right) + \left(3 + 2\frac{H'}{H} + \Gamma\right)\Phi = 0, \quad (25)$$

where we have introduced a dimensionless correction to the equation for standard pressureless dust arising from the perturbation of pressure in our model,

$$\Gamma \equiv \frac{\mu p_\varphi}{2M_{\text{Pl}}^2H'H^2}, \quad (26)$$

with  $(\cdot)' = \partial_N(\cdot)$  the derivative with respect to the number of e-folds. Using the Friedmann equations,  $\Gamma$  can be written in geometrical terms as

$$\Gamma = -\frac{H''}{H'} - \frac{H'}{H} - 3.$$

This particular combination vanishes for a background expansion history mimicking that of  $\Lambda$ CDM. Therefore, in that limit, linear perturbations in our model will behave identically to dust in the presence of a cosmological constant.

Further, similarly to the standard case one can introduce a new variable

$$Q = \sqrt{\frac{a}{-H'}}\Phi,$$

such that Eq. (25) takes the form of an oscillator with time-dependent frequency

$$Q'' - \left(\frac{\Theta''}{\Theta}\right)Q = 0, \text{ where } \Theta \equiv \frac{H}{\sqrt{-H'}}. \quad (27)$$

Our variable  $Q$  is a redefinition  $Q \propto u\sqrt{aH}$  of the standard variable

$$u \propto \frac{\Phi}{\sqrt{-H}} \propto \frac{\Phi}{\sqrt{\varepsilon + p}}, \quad (28)$$

given in [36, p. 302, Eq. (7.63)]. This redefinition is caused by our choice of the time coordinate. The equation of motion for  $u$  is

$$\partial_\eta^2 u - \left(\frac{\partial_\eta^2\theta}{\theta}\right)u = 0, \quad (29)$$

where  $\eta$  is conformal time ( $ad\eta = dt$ ) and

$$\theta \propto a^{-1}(1+w_X)^{-1/2} \propto \frac{1}{a\sqrt{-H}} \propto \frac{\Theta}{\sqrt{aH}}. \quad (30)$$

It is easy to check that equations (27) and (29) are equivalent. Thus, cosmological perturbations of the  $\lambda\varphi$ -fluid reproduce the standard result for hydrodynamical matter in the limit  $c_s = 0$ . We could have guessed the equation of motion (29) and the variable  $u$  from the very beginning because neither the final expression for  $u$  nor the formula for  $\theta$  explicitly involve the sound speed. However, note that the derivation of these results presented in [36, p. 302] uses in an essential way the standard hydrodynamical formula for the sound speed  $c_s^2 = \dot{p}/\dot{\varepsilon}$  which is clearly absolutely inapplicable for the  $\lambda\varphi$ -fluid. Curiously,  $u$  and  $\theta$  are given by the same formulae (29) and (30) in the case of k-essence where the derivation uses a sound speed given by  $c_s^2 = p_X/\varepsilon_X \neq \dot{p}/\dot{\varepsilon}$  (see [36, 37]).

From equation (27), one can see that one of its solutions is  $Q \propto \Theta$ , which translates to one of the modes for the Newtonian potential

$$\Phi \propto \frac{H}{a}.$$

Taking the derivative gives the “instantaneous power law”:

$$\left( \ln \frac{H}{a} \right)' = - \left( \frac{5 + 3w_X}{2} \right),$$

demonstrating that for  $w_X > -5/3$  this solution represents the decaying mode. Following the discussion in [36, p. 303], we find the second mode of the solution from the Wronskian. Integrating this result by parts allows us to write the full solution as

$$\Phi = C_1(\mathbf{x}) + \frac{H}{a} C_2(\mathbf{x}) - C_1(\mathbf{x}) \frac{H}{a} \int^a \frac{da}{H}. \quad (31)$$

This solution is valid on all scales, whereas in the standard case when  $c_s \neq 0$  it is only applicable on superhorizon modes.

### 1. Perturbations Around Scaling Solutions

Let us analyse this general solution further in the case of a cosmology dominated by a  $\lambda\varphi$ -fluid with a constant equation-of-state parameter  $w_X \neq -1$  (we demonstrate how to construct one such class of models in section IV C). In such a case, the final term in the expression above is constant, and our solution for  $w_X \neq -5/3$  is

$$\Phi = \tilde{C}_1(\mathbf{x}) + C_2(\mathbf{x}) a^{-(5+3w_X)/2}, \quad (32)$$

while for the special case of  $w_X = -5/3$  the solution is

$$\Phi_{-5/3} = \tilde{C}_1(\mathbf{x}) + \tilde{C}_2(\mathbf{x}) \ln a.$$

Thus, for all constant  $w_X > -5/3$  the Newtonian potential is constant up to a decaying mode. Substituting this solution (32) into the Poisson equation (17), we obtain for the density fluctuations in the cases  $w_X \neq -1$  and  $w_X \neq -5/3$ :

$$\begin{aligned} \delta_\varepsilon &= -2\Phi + \frac{2}{3} \frac{\Delta}{(aH)^2} \Phi - 2\Phi' = \\ &= -2\tilde{C}_1 + 3\tilde{C}_2(1+w_X) a^{-(5+3w_X)/2} + \\ &\quad + \frac{2}{3} \frac{\Delta}{(aH)^2} (\tilde{C}_1 + \tilde{C}_2 a^{-(5+3w_X)/2}). \end{aligned}$$

For subhorizon modes,  $k \gg aH$ , we have

$$\begin{aligned} (\delta_\varepsilon)_{k \gg aH} &\sim a^{1+3w_X} & \text{for } w_X > -\frac{5}{3}, \\ (\delta_\varepsilon)_{k \gg aH} &\sim a^{3(w_X-1)/2} & \text{for } w_X < -\frac{5}{3}, \end{aligned}$$

while for superhorizon modes,  $k \ll aH$ , we have

$$\begin{aligned} (\delta_\varepsilon)_{k \ll aH} &\sim \text{const} & \text{for } w_X > -\frac{5}{3}, \\ (\delta_\varepsilon)_{k \ll aH} &\sim a^{-(5+3w_X)/2} & \text{for } w_X < -\frac{5}{3}. \end{aligned}$$

It is illustrative to compare these results with those for standard cosmological fluids. In particular, for the ultra-relativistic equation of state,  $w_X = 1/3$ , the Newtonian potential for the  $\lambda\varphi$ -fluid remains constant up to a decaying mode at all scales, while subhorizon density perturbations grow as  $\delta_\varepsilon \sim a^2$ . This should be contrasted with standard radiation which causes both the potential and the density perturbation to decay and oscillate once the mode becomes subhorizon.

### 2. Phantom Behaviour and $w = -1$ Crossing

As the general solution (31) implies, the perturbations do not have catastrophic instabilities even for phantom [38] equations of state, with  $w_X < -1$ . The  $\lambda\varphi$ -fluid framework allows one to realise such scenarios easily. However, Eq. (26) implies that  $\Gamma$  has a singularity when  $H' = 0$ , i.e. when  $w_X = -1$ . Can a  $\lambda\varphi$ -fluid evolve through this singularity, from a standard to a phantom equation of state?

The answer is provided by the analysis of Eq. (25) in the vicinity of the singularity. The derivatives of  $\Phi$  can only remain finite, if the singular terms in the equation cancel, i.e. if  $\Phi' + \Phi = 0$  at the phantom-divide-crossing point. Indeed, it can easily be checked that the general solution (31) always satisfies

$$\Phi' + \Phi = \frac{H'}{a} \left( C_2(\mathbf{x}) - C_1(\mathbf{x}) \int^a \frac{da}{H} \right),$$

so that automatically  $\Phi' + \Phi = 0$  when  $H' = 0$ . Therefore, we have also shown that in our model there are no classical catastrophic instabilities associated with the crossing of the phantom divide.

Note that the  $\lambda\varphi$ -fluid contains only one degree of freedom but the action cannot be written exclusively in terms of this degree of freedom in a generally covariant local form. Therefore this possibility of smooth crossing of the  $w_X = -1$  barrier does not contradict to the statement proved in [39] and rederived in different ways later in [40–46]. The  $\lambda\varphi$ -fluid provides a working example of the so-called Quintom scenario from [47], see also reviews [48–50]. Further, abandoning another assumption from [39], that coupling to gravity is minimal, allows one to have a classically stable crossing of the phantom divide in scalar-tensor theories [51–55]. For other single-field options see [56–58]. For more on phantoms see e.g. [8, 59, 60] and references therein.

One can obtain the equation of motion (29) from the action

$$S_u = \frac{1}{2} \int d\eta d^3x \left( (\partial_\eta u)^2 + \left( \frac{\partial_\eta^2 \theta}{\theta} \right) u^2 \right). \quad (33)$$

Both quantities  $u$  and  $\theta$  (or  $Q$  and  $\Theta$ ) are defined up to a constant factor. In particular, this factor can be a complex number e.g. the imaginary unit  $i$ . Note that

the definitions (28), (30) which we have used imply that  $u$  and  $\theta$  are real, provided that the Null Energy Condition for the background holds i.e.  $\dot{H} < 0$ . As usual, the sign of the action (33) is such that for  $\dot{H} < 0$  it has a positive definite kinetic term. When  $\dot{H} = 0$ , both quantities  $u$  and  $\theta$  diverge. However, as we have already demonstrated, the evolution of the physical quantity  $\Phi$  is smooth through  $\dot{H} = 0$ . This also means that the curvature invariants are smooth through  $w_X = -1$  crossing. Thus after the crossing  $u$  and  $\theta$  become pure imaginary or in terms of real fields, the action changes the overall sign and the kinetic term becomes negative definite.

#### IV. EXAMPLE $\lambda\varphi$ -FLUID COSMOLOGIES

##### A. $\lambda\varphi$ -Dust with Cosmological Constant

Let us consider the simplest case:  $K_\varphi = 0$  and  $\mu = \text{const}$ . In that case, we have

$$p(\mu) = K \quad \text{and} \quad \varepsilon = \mu^2(K_X + \lambda) - K,$$

where  $K(\mu) = \text{const}$  and  $K_X(\mu) = \text{const}$ . For the equations of motion, Eq. (14), we have

$$\begin{aligned} \varphi &= \mu t, \\ \dot{\lambda} &= -3H(K_X + \lambda), \end{aligned}$$

with the solution

$$(K_X + \lambda) = \frac{\mu^{-2}\varepsilon_0}{a^3},$$

where  $\varepsilon_0$  is a constant of integration. Thus the system behaves as

$$p = \text{const} \quad \text{and} \quad \varepsilon = \varepsilon_0 a^{-3} - p,$$

so that the energy-momentum corresponds to a mixture of a cosmological constant  $\Lambda = -K(\mu)$  of either sign and pressureless dust with the energy density  $\varepsilon_0$  today. Since the background is that of  $\Lambda$ CDM cosmology, by the argument of section III B, the evolution of  $\Phi$  and  $\delta_\varepsilon$  also exactly reproduces the standard results.

##### B. $\lambda\varphi$ -Dust

If we take  $K = 0$  and an arbitrary  $\mu(\varphi)$  we obtain a  $\lambda\varphi$ -fluid which mimics pure dust,  $p = 0$ , with energy density  $\varepsilon = \mu^2\lambda$ . Indeed from Eq. (14) we have

$$\begin{aligned} \dot{\varphi} &= \mu(\varphi), \\ \dot{\lambda}\mu^2 &= -2\mu\dot{\mu}\lambda - 3H\varepsilon, \end{aligned}$$

with the standard solution  $\varepsilon = \varepsilon_0 a^{-3}$ . Note that there is a degeneracy in  $\varphi(t)$ . In this setup for different  $\mu(\varphi)$ , the same evolution of the energy density  $\varepsilon(t)$  corresponds to different  $\varphi(t)$ . Again, by our discussion of cosmological perturbations in section III B, the evolution of  $\Phi$  and  $\delta_\varepsilon$  exactly reproduces the results for the standard dust-dominated universe.

##### C. $\lambda\varphi$ -Fluid Possessing a Scaling Solution

Moving beyond a constant  $\mu$ , let us consider a class of models with

$$K = \sigma X, \quad \text{where} \quad \sigma = \pm 1, \quad (34)$$

$$\mu = \mu_0 \exp\left(-\frac{\varphi}{m}\right), \quad (35)$$

where the mass scale for  $\varphi$  is

$$m = \sqrt{\frac{8}{3}} \frac{\sqrt{\sigma w_{\text{fin}}}}{1 + w_{\text{fin}}} M_{\text{Pl}}. \quad (36)$$

In the following, we will show that, in this class of models, the dynamics of the  $\lambda\varphi$ -fluid-dominated cosmological background has a fixed point with a constant equation of state  $w_{\text{fin}}$ , where  $w_{\text{fin}}$  can have either sign and can even be phantom-like.<sup>4</sup> Further, we will show that this fixed point solution is an attractor provided  $w_{\text{fin}} < 1$ .

From Eqs (8), (7) we obtain

$$p = \frac{\sigma}{2}\mu^2 \quad \text{and} \quad \varepsilon = \mu^2\left(\frac{\sigma}{2} + \lambda\right), \quad (37)$$

so that the *instantaneous* equation of state,

$$w_X = \frac{1}{1 + 2\sigma\lambda}, \quad (38)$$

is determined by the value of  $\lambda$ . From this equation it follows that, if  $w_X = \text{const}$ , then

$$\lambda_{w_X} = \frac{1}{2}\sigma(w_X^{-1} - 1) = \text{const}. \quad (39)$$

In particular, for values of  $w_X$  corresponding to an accelerating expansion,  $\lambda_{w_X}$  is a number of order one. Meanwhile, the exponential form of  $\mu(\varphi)$ , Eq. (35) implies that the evolution of  $\mu$  has a very simple form,

$$\mu = \frac{m}{t} \quad \text{and} \quad \mu_\varphi = -\frac{\mu}{m} = -\frac{1}{t}, \quad (40)$$

where we have chosen constants of integration in such a way that the pressure  $p$  of the  $\lambda\varphi$ -fluid is singular exactly at the Big Bang,  $t = 0$ .

We can now rewrite the equation of motion Eq. (14) combined with the Friedmann equation for a universe containing solely the  $\lambda\varphi$ -fluid, to obtain the equation of motion for the equation of state

$$w'_X = 3w_X \left(1 + w_X - \sqrt{\frac{w_X}{w_{\text{fin}}}(1 + w_{\text{fin}})}\right). \quad (41)$$

It is easy to see that  $w_{\text{fin}}$  is the fixed point of this equation, and therefore also of the equation of motion for  $\lambda$ . In the limit of  $w_{\text{fin}} \rightarrow -1$ , this equation reduces to the

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<sup>4</sup> Obviously we cannot realise  $w_{\text{fin}} = 0$  in this setup.

evolution of  $w$  for  $\Lambda$ CDM. Therefore, as  $w_{\text{fin}}$  approaches that limit, the transition from matter domination to dark energy domination becomes indistinguishable from that in  $\Lambda$ CDM.

Let us now consider the stability of this fixed point. Linearising the above around  $w_X = w_{\text{fin}}$ , we obtain

$$\delta w'_X = \frac{3}{2}(w_{\text{fin}} - 1)\delta w_X.$$

From this result, it follows that in an expanding universe, for  $w_{\text{fin}} < 1$ , the equation of state approaches  $w_{\text{fin}}$  and  $\lambda$  approaches the fixed point  $\lambda_{w_{\text{fin}}}$ . We call this solution the  $w$ -attractor.

Eq. (41) can actually be solved, albeit implicitly, allowing us to obtain the scale factor  $a$  as a function of the equation of state parameter  $w_X$ :

$$\left(\frac{a}{a_0}\right)^{3(w_{\text{fin}}-1)} = \left[\frac{\left(\sqrt{w_{\text{fin}}/w_X} - w_{\text{fin}}\right)^{w_{\text{fin}}}}{\sqrt{w_{\text{fin}}/w_X} - 1}\right]^2, \quad (42)$$

where  $a_0$  is a constant of integration.

It is instructive to compare the evolution of the  $\lambda\varphi$ -fluid given by Eq. (42) or Eq. (41) with the case of a mixture of dust and DE with a constant equation of state  $w = w_{\text{fin}}$ . For  $w_{\text{fin}} < 0$  the evolution of the universe has the same late-time asymptotic as in the  $\lambda\varphi$ -fluid case. From the Friedmann equations (15) and (16) we obtain

$$w'_{\text{dark}} = 3w_{\text{dark}}(w_{\text{dark}} - w_{\text{fin}}), \quad (43)$$

instead of Eq. (41), where we have denoted the total effective equation of state as  $w_{\text{dark}}$ . These two equations (41) and (43) only coincide in the limit of  $w_{\text{fin}} \rightarrow -1$ . For illustrative purposes we also present the solution for (43) in the form similar to Eq. (42):

$$\left(\frac{a}{a_0}\right)^{3w_{\text{fin}}} = \frac{w_{\text{fin}}}{w_{\text{dark}}} - 1.$$

From this analysis then, the evolution history of the  $\lambda\varphi$ -fluid cannot be reduced to the evolution of a mixture of DE with a constant equation of state parameter  $w_{\text{fin}}$  and dust, providing a potential observational probe for such theories.

In the following, we present cosmologies in the two limits where the  $\lambda\varphi$ -fluid dominates the total energy density of the universe and when it is subdominant to some other matter fluid.

### 1. Dominant $\lambda\varphi$ -Fluid in FRW Universe

Let us again consider the model given by Eqs (37) and (34), however, this time in the case with  $\lambda \gg 1$ , far off the  $w$ -attractor. Here, the system is effectively pressureless, by virtue of Eq. (38). The energy density scales as  $a^{-3}$  since  $w_X \approx 0$ . The evolution off the attractor

therefore resembles a matter-dominated epoch, eventually approaching an era with a constant  $w_X = w_{\text{fin}}$ .

Approximating Eq. (41) for small  $w_X$  we can calculate how the equation of state evolves during this period:

$$w'_X \simeq 3w_X \Rightarrow w_X \sim a^3 \Rightarrow \lambda \sim a^{-3}.$$

Turning to perturbations during this matter-domination era, the pressure corrections Eq. (26) can be written as

$$\Gamma(w_X) = 3w_X \sqrt{\frac{w_X}{w_{\text{fin}}}} \left(\frac{1+w_{\text{fin}}}{1+w_X}\right) \sim a^{-9/2}. \quad (44)$$

where we can explicitly see that  $\Gamma \rightarrow 0$  as  $w_{\text{fin}} \rightarrow -1$ . For parameter values motivated by dark energy,  $|1+w_{\text{fin}}| \ll 1$ , these pressure corrections to the evolution of the potential  $\Phi$  are subleading until  $w_X$  approaches its attractor value, since in Eq. (25)

$$\Phi'' + \Phi' \left(\frac{5}{2} - \frac{3}{2}w_X + \Gamma\right) + (-3w_X + \Gamma)\Phi = 0. \quad (45)$$

So, while off the attractor, there is no significant deviation in the growth of structure from standard considerations.

On the attractor,  $w_X = w_{\text{fin}}$ , we obtain

$$\Gamma = 3w_{\text{fin}},$$

and Eq. (45) confirms what we found in section III B for the general model: for  $w_{\text{fin}} > -5/3$ ,  $\Phi = \text{const}$ , while  $\delta_\varepsilon \sim a^{1+3w_{\text{fin}}}$  is the growing mode inside the horizon.

### 2. Subdominant $\lambda\varphi$ -Fluid in FRW Universe

Motivated by the discussion above let us still consider theories with

$$\mu = \mu_0 \exp\left(-\frac{\varphi}{m}\right),$$

where  $m = \text{const}$  and may, for example, be given by Eq. (36). Using the result of Eq. (40) we obtain the equation of motion for  $\lambda$ ,

$$\dot{\lambda} = \frac{1}{t}(\sigma + 2\lambda) - 3H(\sigma + \lambda). \quad (46)$$

In the history of the universe there were at least two stages with the background equation of state,  $w_b$ , approximately constant, namely the radiation-dominated epoch—with  $w_b \simeq \frac{1}{3}$ —and the matter-dominated era—with  $w_b \simeq 0$ . Let us then consider the dynamics of Eq. (46) when the matter content of universe consists mostly of some fluid with a constant equation of state parameter,  $w_b = \text{const}$ . In that case, we have

$$a = \left(\frac{t}{t_0}\right)^{2/3(w_b+1)} \quad \text{and consequently} \quad H = \frac{2}{3(w_b+1)t},$$

here  $t_0$  would correspond to the age of the universe today, when  $a = 1$ , if  $w_b = \text{const}$ . On this background, Eq. (46) for the subdominant  $\lambda\varphi$ -fluid can be expressed as

$$\frac{d\lambda}{d\ln t} = \sigma \frac{w_b - 1}{w_b + 1} + \frac{2w_b}{w_b + 1} \lambda. \quad (47)$$

This equation has the following solution

$$\begin{aligned} \lambda(t) &= \lambda_{w_b} + Ct^{2w_b/(w_b+1)} = \lambda_{w_b} + Ca^{3w_b}, & \text{for } w_b \neq 0, \\ \lambda(t) &= -\ln\left(\frac{t}{t_1}\right), & \text{for } w_b = 0, \end{aligned}$$

where  $\lambda_w$  is the fixed point solution given by the formula (39) and  $C$  and  $t_1$  are constants of integration. We can therefore see that, for  $w_b \neq 0$ , the  $\lambda\varphi$ -fluid corresponds to a *mixture* of a fluid with the same equation of state  $w_b$  as the background and, in addition, dust

$$\begin{aligned} p &= \frac{\sigma}{2} m^2 t^{-2} = \frac{\sigma}{2} \left(\frac{m}{t_0}\right)^2 a^{-3(1+w_b)}, \\ \varepsilon &= w_b^{-1} p + \left(\frac{m}{t_0}\right)^2 C a^{-3}, \end{aligned}$$

while for a dust-like background,  $w_b = 0$ , the  $\lambda\varphi$ -fluid's hydrodynamics obeys

$$\begin{aligned} p &= \frac{\sigma}{2} m^2 t^{-2} = \frac{\sigma}{2} \left(\frac{m}{t_0}\right)^2 a^{-3}, \\ \varepsilon &= p - \frac{m^2}{t^2} \ln\left(\frac{t}{t_1}\right) = p \left(1 - 2\sigma \ln\left(\frac{t}{t_1}\right)\right). \end{aligned}$$

In the late-time asymptotic, the equation of state of the  $\lambda\varphi$ -fluid approaches that of the background provided that  $w_b < 0$ . In particular, this tracking behaviour means that in an inflationary background when  $w_b \simeq -1$ , the  $\lambda\varphi$ -fluid does not redshift away but instead survives in form of an effective cosmological constant.

## V. DUSTY DARK ENERGY

In this section, we will present a unified dark matter and dark energy model using a single dynamical degree of freedom, which we call *Dusty Dark Energy*.

The discussion of the constant- $w$  model presented in section IV C is suggestive: a model with  $\mu(\varphi)$  of the form given in Eq. (35) has a background evolution that evolves as dust when it is far off its attractor and eventually settles on a constant equation of state determined by the value of the mass-scale parameter  $m$ . This picture is not altered when the  $\lambda\varphi$ -fluid is subdominant during a radiation-domination epoch: the dust-like component will eventually overwhelm the redshifting radiation resulting in a period of expansion similar to matter domination with its duration determined by the value of  $\lambda$  when radiation becomes subdominant. Once  $\lambda$  approaches a

number of order unity, the transition to dark-energy domination will occur at a rate similar to the usual transition in  $\Lambda$ CDM and eventually the expansion will settle on a constant (and arbitrary) equation of state. This background evolution provides an expansion history that is close to  $\Lambda$ CDM and yet does not explicitly contain separate dark-matter and dark-energy components: only radiation and our  $\lambda\varphi$ -fluid are necessary.

Beyond describing the background dynamics, we have to show that the perturbations in the fluid evolve similarly to those in  $\Lambda$ CDM. This is necessary not only for the growth of large-scale structure during matter domination, but also during the radiation-domination epoch: the anisotropies in the cosmic microwave background depend on the existence of appropriate gravitational potentials driven by dark matter.

In section IV C, we have shown that during matter domination and at late times perturbations evolve similarly to the CDM perturbations in  $\Lambda$ CDM. The remaining piece is to prove that the perturbations evolve in a dust-like manner even when the energy density is dominated by radiation, which we will show as follows.

During domination by radiation—or any external fluid in general—the Newtonian potential,  $\Phi$ , is driven by perturbations in that fluid. The subdominant components then respond to this potential and evolve according to equations of motion arising from the conservation of their EMT. For dust, the standard result, written in a suggestive manner, is

$$\begin{aligned} \left(\frac{\delta\varphi_{\text{dust}}}{\mu}\right)^\cdot &= \Phi, \\ \dot{\delta}_{\text{dust}} &= 3\dot{\Phi} + \frac{\Delta}{a^2} \left(\frac{\delta\varphi_{\text{dust}}}{\mu}\right). \end{aligned} \quad (48)$$

In the case of a subdominant Dusty Dark Energy, the equivalent equations, Eq. (20) and (22) tell us that the eventual evolution to a matter-dominated era requires that  $\lambda \gg 1$ , implying  $w_X \sim 1/2\lambda$ . Therefore we can approximate

$$\left(\frac{\delta\varphi}{\mu}\right)^\cdot = \Phi, \quad (49)$$

$$\begin{aligned} \dot{\delta}_\varepsilon &\simeq 3\dot{\Phi} + \frac{\Delta}{a^2} \left(\frac{\delta\varphi}{\mu}\right) - \\ &- 3Hm^{-1}\lambda^{-1}\delta\varphi - \frac{3}{2}H\lambda^{-1}\delta_\varepsilon. \end{aligned} \quad (50)$$

All we now need to do is to show that the terms additional to the equations for CDM are negligible. For  $\lambda \gg 1$ ,

$$\dot{\delta}_\varepsilon \sim H\delta_\varepsilon \gg H\lambda^{-1}\delta_\varepsilon.$$

From Eq. (49) we can obtain the estimate

$$\delta\varphi \sim \frac{\mu\Phi}{H}.$$

Then, the pressure correction in Eq. (50) is small compared to the Newtonian potential providing

$$\dot{\Phi} \gtrsim \Phi H \gg \mu\Phi m^{-1}\lambda^{-1},$$

or, in other words, if

$$1 \gg \frac{\mu}{mH\lambda}. \quad (51)$$

On the other hand

$$\frac{\mu}{mH} \sim \frac{(1 + w_{\text{fin}}) \sqrt{-p_X}}{HM_{\text{Pl}}} \sim (1 + w_{\text{fin}}) \sqrt{\frac{-p_X}{p_{\text{rad}}}}.$$

Since we assume that the DDE is subdominant,  $\varepsilon_{\text{rad}} = 3p_{\text{rad}} \gg |p_X|$  and, by construction,  $|1 + w_{\text{fin}}| \ll 1$ . Therefore all the additional terms in the evolution equations are negligible, and the perturbations in the DDE obey the same equations for evolution as standard dust during radiation domination, demonstrating that the DDE model provides for a viable cosmology with our fluid playing the role of *both* cold dark matter and dark energy.

The duration of the matter-domination era is determined by the initial values of  $\lambda(t_0) \gg 1$  for any given initial  $\varphi(t_0)$ . The former determines the time when the DDE turns over from dark-matter-like to dark-energy-like, while the latter sets the time where it begins to dominate over radiation. Since we have to tune both values, we have not provided a solution to the coincidence problem. On the other hand, this model is a “minimalist” description—as we require only two initial conditions for set the transitions of the two epochs.

## VI. SUMMARY

In this paper, we introduced a novel class of field theories with a single dynamical degree of freedom which have a perfect-fluid interpretation. The key feature of this theory is that its fluid velocity flows along geodesics—hence mimicking “dust” in this respect. On the other hand, unlike a standard cold-dark-matter fluid, it carries pressure parallel to its fluid velocity. In cosmology, this pressure affects the expansion history.

This sleight-of-hand is achieved by means of a “Lagrange multiplier” field, employed to constrain by equation of motion the above-mentioned behaviour of the fluid velocity vector. Our system then consists of *two first order equations of motion*, and hence effectively a single degree of freedom. This dynamic cannot be reproduced by usual scalar field theories such as *k-essence* or higher derivative theories.

As an application, we consider the evolution and effects of this fluid in cosmology. We show that there exists a class of scaling solutions which have an attractor solution with fixed equation of state  $w_{\text{fin}}$ . Off the attractor, this model possess an interesting dynamic where part of the energy density redshifts as dust while part of the energy density tracks any dominant background energy density. We use this curious property to construct a unified dark energy/dark matter model where the limit  $w_{\text{fin}} = -1$  corresponds to standard  $\Lambda$ CDM.

We also show that in this class of models, we can construct phantom models with  $w_{\text{fin}} < -1$  where there is no pathology when crossing the “phantom divide” at  $w = -1$ , at least classically. If we insist that the system satisfy the Null Energy Condition (i.e.  $w \geq -1$ ), then we show that the kinetic term for perturbations is positive definite.

Finally, we would like to conclude by emphasising that this class of theories provides a novel framework for cosmological model building and exploring exotic states of matter.

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## Appendix A: Derivation of perturbed energy conservation

The differential form of the perturbed constraint equation (20) is

$$\delta\ddot{\varphi} = (\mu_\varphi^2 + \mu\mu_{\varphi\varphi})\delta\varphi + \dot{\Phi}\mu + 2\Phi\mu\mu_\varphi. \quad (A1)$$

Let us perturb the energy-conservation equation (9):

$$\delta u^\mu \nabla_\mu \varepsilon + \delta\dot{\varepsilon} + \delta\theta(\varepsilon + p) + 3H(\delta\varepsilon + \delta p) = 0. \quad (A2)$$

First of all, as usual we have  $\delta u^t = -\Phi$  so that

$$\delta u^\mu \nabla_\mu \varepsilon = \dot{\varepsilon}\delta u^t = -\Phi\dot{\varepsilon}. \quad (A3)$$

Further, we perturb the formula for the expansion Eq. (10)

$$\delta\theta = \mu^{-1}\delta(\square\varphi) - (\mu^{-2}\square\varphi\mu_\varphi + \mu_{\varphi\varphi})\delta\varphi.$$

For the perturbations of the d’Alambertian we have the standard result

$$\delta(\square\varphi) = -4\dot{\Phi}\dot{\varphi} - 2\Phi\ddot{\varphi} - 6H\Phi\dot{\varphi} + \delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\Delta}{a^2}\delta\varphi,$$

which after using Eqs (11) and (20) can be written in terms of  $\Phi$  and  $\delta\varphi$  as

$$\delta(\square\varphi) = \left( \mu_\varphi^2 + \mu\mu_{\varphi\varphi} + 3H\mu_\varphi - \frac{\Delta}{a^2} \right) \delta\varphi - 3\mu(\dot{\Phi} + H\Phi).$$

Therefore for the perturbation of the expansion we obtain

$$\delta\theta = -3 \left( \dot{\Phi} + H\Phi \right) - \mu^{-1} \frac{\Delta}{a^2} \delta\varphi.$$

Substituting this expression along with along the formula (A3) into Eq. (A2) results in

$$\delta\dot{\varepsilon} - \left( 3\dot{\Phi} + \frac{\Delta}{a^2} \left( \frac{\delta\varphi}{\mu} \right) \right) (\varepsilon + p) + 3H(\delta\varepsilon + \delta p) = 0,$$

which corresponds to the standard case (c.f. [36, p. 312, Eq. (7.105)]).

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